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A dependence domain for a class of microdifferential operators with involutive double characteristics

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In the study of the mathematical models for conical refraction, the theory of 2-microlocal analysis plays an important role. We give, in this note, a sharp microlocal dependence domain for a class of microdifferential operators, which is also an application of 2-microlocal analysis.

1 Statement of the Main Theorem

Let M be a real analytic manifold with complexification X , and P a microdifferential operator defined in a neighborhood U in T^*X of a point $\dot{q} \in T_M^*X \setminus M$. We assume that the characteristic variety of P satisfies

$$\text{Char}(P) \subset \{q \in U; p_1(q) = p_2(q) = 0\}$$

with homogeneous holomorphic functions p_1 and p_2 on U with the following properties:

- p_1 and p_2 are real valued on T_M^*X ,
- $dp_1 \wedge dp_2 \wedge \omega_X(q) \neq 0$ if $p_1(q) = p_2(q) = 0$,
- $\{p_1, p_2\}(q) = 0$ if $p_1(q) = p_2(q) = 0$.

Here ω_X is the canonical 1-form of T^*X , and $\{\cdot, \cdot\}$ the Poisson bracket on T^*X .

In this situation, we can define regular involutive submanifolds $V^{\mathbb{C}} \subset T^*X$ and $V \subset T_M^*X$ by

$$\begin{aligned} V^{\mathbb{C}} &= \{q \in U; p_1(q) = p_2(q) = 0\}, \\ V &= V^{\mathbb{C}} \cap T_M^*X, \end{aligned}$$

and we assume, for the simplicity, that $\dot{q} \in V$. Moreover Γ denotes the canonical leaf of V passing through \dot{q} .

A set $K \subset \Gamma$ is called a Γ -rectangle if there exists an injective real analytic map

$$\Phi : [0, 1] \times [0, 1] \longrightarrow \Gamma$$

with the following three properties:

- $\Phi([0, 1] \times [0, 1]) = K$
- $\Phi(\cdot, t)$ is an integral curve of the Hamiltonian vector field H_{p_1} for any fixed $t \in [0, 1]$.
- $\Phi(s, \cdot)$ is an integral curve of the Hamiltonian vector field H_{p_2} for any fixed $s \in [0, 1]$.

We give, in this situation,

Theorem 1.1. *There exists an open neighborhood U_0 of \dot{q} in Γ with the property that for any Γ -rectangle K contained in U_0 with the four vertices q_0, q_1, q_2 , and q_3 and for any microfunction solution u to $Pu = 0$ on K ,*

$$q_1, q_2, q_3 \notin \text{supp}(u) \implies q_0 \notin \text{supp}(u).$$

This theorem can be deduced from the model case given in the next section.

2 Theorem in the model case

Let M be an open subset of \mathbb{R}^n with a complex neighborhood X in \mathbb{C}^n ($n \geq 3$). We take a coordinate system of M (resp. X) as $x = (x_1, \dots, x_n)$ (resp. $z = (z_1, \dots, z_n)$). Then $(x; \sqrt{-1}\xi \cdot dx)$ (resp. $(z; \zeta \cdot dz)$) denotes a point in T_M^*X (resp. T^*X) with $\xi = (\xi_1, \dots, \xi_n)$ (resp. $\zeta = (\zeta_1, \dots, \zeta_n)$).

We take a point $q_0 = (0; \sqrt{-1}dx_n) \in T_M^*X$. Let P be a microdifferential operator defined in a neighborhood of q_0 whose principal symbol is of the form

$$\zeta_1^{m_1} \zeta_2^{m_2}$$

with $m_1, m_2 \geq 1$. We define an involutive manifold V of T_M^*X by

$$V = \{(x; \sqrt{-1}\xi \cdot dx); \xi_1 = \xi_2 = 0\}$$

and denote by Γ the leaf of V passing through the point q_0 . We take a rectangle K on Γ defined by

$$K = \{(x_1, x_2, x'' = 0; \sqrt{-1}dx_n); 0 \leq x_1 \leq t_1, \quad 0 \leq x_2 \leq t_2\}.$$

Here $x'' = (x_3, \dots, x_n)$. The vertices of K are denoted by

$$q_0, q_1 = (t_1, 0, 0; \sqrt{-1}dx_n), q_2 = (0, t_2, 0; \sqrt{-1}dx_n), q_3 = (t_1, t_2, 0; \sqrt{-1}dx_n).$$

Then we have

Theorem 2.1. *Let u be a microfunction defined in a neighborhood of K . We assume that u satisfies*

$$Pu = 0$$

and that the three points q_1, q_2, q_3 are not in $\text{supp}(u)$:

$$q_1, q_2, q_3 \notin \text{supp}(u).$$

Then

$$q_0 \notin \text{supp}(u).$$

Remark 2.2. The phenomenon in Theorem 2.1 was first observed by Y. Okada [O] for C^∞ wavefront set of microdistribution solutions. His result concerns with the case $m_1 = m_2 = 1$ under a Levi condition on the lower order term of P . He employed a microlocal version of Goursat problem in the complex domain.

To give an implication of Theorem 2.1, we recall a result obtained by N. Tose [T2].

Theorem 2.3. *Let u be a microfunction solution to $Pu = 0$ on an open subset U of Γ . Then there exist a family $\{b_\lambda^{(1)}\}_{\lambda \in \Lambda_1}$ of integral curves on Γ of $\partial/\partial x_1$ and another family $\{b_\lambda^{(2)}\}_{\lambda \in \Lambda_2}$ of integral curves on Γ of $\partial/\partial x_2$ which satisfy the properties that the set*

$$\bigcup_{\lambda \in \Lambda_1} b_\lambda^{(1)} \cup \bigcup_{\lambda \in \Lambda_2} b_\lambda^{(2)}$$

is included in $\text{supp}(u)$ and that $\text{supp}(u)$ has unique continuation property on the set

$$\Omega = U \setminus \left(\bigcup_{\lambda \in \Lambda_1} b_\lambda^{(1)} \cup \bigcup_{\lambda \in \Lambda_2} b_\lambda^{(2)} \right).$$

More precisely, if a point $q \in \Omega$ is not in $\text{supp}(u)$, then the connected component of Ω containing q is disjoint with $\text{supp}(u)$.

In the situation of Theorem 2.3, we take a point

$$\dot{q} = (s_1, s_2, x'' = 0; \sqrt{-1}dx_n) \in \Gamma.$$

We assume that, for a neighborhood U_1 of \dot{q} , the only one integral curve $b_{\lambda_1}^{(1)}$ of $\partial/\partial x_1$ and the only one $b_{\lambda_2}^{(2)}$ of $\partial/\partial x_2$ pass U_1 . We assume, for simplicity, that the both two curves pass \dot{q} :

$$\dot{q} \in b_{\lambda_j}^{(j)} \quad (j = 1, 2).$$

We assume that

$$\text{supp}(u) \cap U_1 \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 < s_1, x_2 > s_2\} = \emptyset$$

and that

$$\text{supp}(u) \cap U_1 \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 > s_1, x_2 < s_2\} = \emptyset.$$

In this situation, if a point

$$\dot{q}' \in \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 < s_1, x_2 < s_2\}$$

does not belong to $\text{supp}(u)$, then it follows from Theorem 2.1 that

$$\text{supp}(u) \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n); x_1 > s_1, x_2 > s_2\} = \emptyset.$$

We also give the following decomposition proposition.

Proposition 2.4. *Let $u \in \mathcal{C}_M(K)$ be a solution to $Pu = 0$. Then there exist two microfunctions u_1 and u_2 in $\mathcal{C}_M(K)$ with the properties*

- $u = u_1 + u_2$,
- $Pu_j = 0$ ($j = 1, 2$),
- $\text{SS}_V^2(u_j) \setminus V \subset \{(x; \sqrt{-1}\xi'' \cdot dx''; \sqrt{-1}(x_1^* \cdot dx_1 + x_2^* \cdot dx_2)); x_j^* = 0\}$.

Here $\text{SS}_V^2(\cdot)$ is the second singular spectrum along V . Using this decomposition, we can prove the Theorem 2.1, but we omit the details.

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